Tensor

1. Distinguish between a null vector and a zero vector in a Riemannian Space.

$\Rightarrow$ In a ~~con~~ Riemannian space a Contravariant vector A^i is said to be a null vector if

$$g_{ij} A^i A^j = 0.$$

Also, a covariant vector is said to be a null vector if.

$$g^{ij} A_i A_j = 0.$$

Again, A vector whose components are all zero in a coordinate system is called a zero vector.

$\therefore$ The components of a zero vector are all zero, it follows that zero vectors are null vectors. But the converse is not necessarily true.

2. If $\psi(x^1, x^2, \ldots, x^n)$ is a scalar, test whether $\dfrac{\partial^2 \psi}{\partial x^i \partial x^j}$ is a Tensor.

$\Rightarrow$ Since ψ is , we have,

$$\bar{\psi} = \psi.$$

Then, $$\frac{\partial \bar{\psi}}{\partial \bar{x}^j} = \frac{\partial \psi}{\partial x^p} \cdot \frac{\partial x^p}{\partial \bar{x}^j}, \quad \text{since } \bar{\psi} = \psi$$

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial^2 \psi}{\partial x^l \cdot \partial x^p} \cdot \frac{\partial x^l \cdot \partial x^p}{\partial \bar{x}^i \cdot \partial \bar{x}^j} + \frac{\partial \psi}{\partial x^p} \cdot \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}$$

$$= \frac{\partial^2 f}{\partial x^l \partial x^p} \cdot \frac{\partial x^l}{\partial \bar{x}^i} \cdot \frac{\partial x^p}{\partial \bar{x}^j} + \frac{\partial \psi}{\partial x^p} \cdot \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}$$

Due to the presence of the second term in the right hand side, $\frac{\partial^2 \phi}{\partial x^i \partial x^j}$ is not a tensor in general.

3. If A^i and B_i are a contravariant vector and covariant vector respectively, show that the sum $A^i B_i$ is an invariant.

$\Rightarrow$ By the transformation law of covariant and contravariant vectors we have,

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^k} A^k$$

and $$\bar{B}_i = \frac{\partial x^j}{\partial \bar{x}^i} B_j$$

Hence $$\bar{A}^i \bar{B}_i = \frac{\partial \bar{x}^i}{\partial x^k} A^k . \frac{\partial x^j}{\partial \bar{x}^i} B_j$$

$$= \frac{\partial \bar{x}^i}{\partial x^k} . \frac{\partial x^j}{\partial \bar{x}^i} A^k B_j$$

$$= \delta^j_k A^k B_j$$

$$= A^j B_j$$

$$= A^i B_i$$

changing the dummy index j by i.

This means that $A^i B_i$ is an invariant.

4. If A^i, B^j are arbitrary contravariant tensor and $c_{ij} A^i B^j$ is an invariant, show that c_{ij} is a 2nd order covariant tensor.

$\Rightarrow$ $$\bar{c}_{pq} \bar{A}^p \bar{B}^q = c_{ij} A^i B^j$$

$$\bar{c}_{pq} \cdot \frac{\partial \bar{x}^p}{\partial x^i} A^i \frac{\partial \bar{x}^q}{\partial x^j} B^j = c_{ij} A^i B^j$$

or, $$\left(\bar{c}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} - c_{ij}\right) A^i B^j = 0$$

Since $A^i B^j$ are arbitrary tensor,

$$\bar{c}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} = c_{ij}$$

Multiplying both side by $\frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s}$ and summing we get,

$$\bar{c}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s} = c_{ij} \frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s}$$

or, $$\bar{c}_{pq} \frac{\partial \bar{x}^p}{\partial \bar{x}^r} \cdot \frac{\partial \bar{x}^q}{\partial \bar{x}^s} = c_{ij} \frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s}$$

or, $$\bar{c}_{pq} \bar{\delta}^p_r \bar{\delta}^q_s = c_{ij} \frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s}$$

or, $$\bar{c}_{rq} \bar{\delta}^q_s = c_{ij} \frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s}$$

or, $$\bar{c}_{rs} = c_{ij} \frac{\partial x^i}{\partial \bar{x}^r} \cdot \frac{\partial x^j}{\partial \bar{x}^s}$$

This shows that c_{ij} is a covariant tensor of type $(0,2)$.

5. Show that the angle between the vectors $(1,0,0,0)$ and $(\sqrt{2},0,0,\sqrt{3}/c)$, c being constant, in a space with line element given by

$ds^2 = -(+dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 (dx^4)^2$ is not real.

$$\Rightarrow ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 (dx^4)^2$$

$g_{11} = -1, \; g_{22} = -1, \; g_{33} = -1, \; g_{44} = c^2.$

$$g_{ij} A^i A^j = g_{11} A^1 A^1 + g_{22} A^2 A^2 + g_{33} A^3 A^3 + g_{44} A^4 A^4$$
$$= (-1)\cdot(1)\cdot(1) + (-1)\cdot 0 \cdot 0 + (-1)\cdot 0\cdot 0 + c^2\cdot 0 \cdot 0$$
$$= -1$$

$$g_{ij} B^i B^j = g_{11} B^1 B^1 + g_{22} B^2 B^2 + g_{33} B^3 B^3 + g_{44} B^4 B^4.$$
$$= (-1)\cdot\sqrt{2}\cdot(\sqrt{2}) + (-1)\cdot 0\cdot 0 + (-1)\cdot 0\cdot 0 + (+c)^2 \frac{\sqrt{3}}{c}\cdot\frac{\sqrt{3}}{c}.$$
$$= -2 + 3 = 1.$$

$$g_{ij} A^i B^j = g_{11} A^1 B^1 + g_{22} A^2 B^2 + g_{33} A^3 B^3 + g_{44} A^4 B^4$$
$$= (-1)\cdot 1\cdot \sqrt{2} + (-1)\cdot 0\cdot 0 + (-1)\cdot 0\cdot 0 + c^2\cdot 0\cdot \frac{\sqrt{3}}{c}$$
$$= -\sqrt{2}$$

If θ be the angle, then

$$\cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{g_{ij} A^i A^j}\cdot\sqrt{g_{ij} B^i B^j}} = \frac{-\sqrt{2}}{\sqrt{-1}\cdot\sqrt{1}}$$
$$= \frac{-\sqrt{2}}{i} = \frac{i^2}{i}\sqrt{2}$$
$$= \sqrt{2}\, i$$

which is not real.

6. Define an unit vector in a Riemannian space.

=> A covariant vector A_i is said to be a unit vector or a covariant vector of unit length if $g^{ij} A_i A_j = 1$

A contravariant vector A^i is said to be a unit vector if $g_{ij} A^i A^j = 1.$

6. In a Riemannian space V^2, find the quantities g^{ij} if $g_{ij} = i + j$.

$\Rightarrow$ $g_{11} = 1+1 = 2$ | $g_{21} = 2+1 = 3$

$g_{12} = 1+2 = 3$ | $g_{22} = 2+2 = 4$

$$g = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$$

$\therefore g^{11} = \frac{4}{-1} = -4$ | $g^{21} = \frac{-3}{-1} = 3$

$g^{12} = \frac{-3}{-1} = 3$ | $g^{22} = \frac{2}{-1} = -2$

$$\therefore g^{ij} = \begin{vmatrix} -4 & 3 \\ 3 & -2 \end{vmatrix}.$$

7. The components of a contravariant tensor in the x-coordinate system are 8 and 4. Find its components in the $\bar{x}$ coordinate system if $\bar{x}^1 = 3x^1$ and $\bar{x}^2 = 5x^1 + 3x^2$.

$\Rightarrow$ Here $A^1 = 8$, $A^2 = 4$.

we find $\bar{A}^1$ and $\bar{A}^2$.

Law for transformation is

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q \quad . \quad p, q = 1, 2$$

$$\therefore \bar{A}^1 = \frac{\partial \bar{x}^1}{\partial x^q} A^q \quad . \quad q = 1, 2$$

$$= \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2$$

$$= 8.3 + 4.0 = 24$$

$$\bar{A}^2 = \frac{\partial \bar{x}^2}{\partial x^1} A^1 + \frac{\partial \bar{x}^2}{\partial x^2} A^2$$

$$= 8 \cdot 5 + 4 \cdot 3 = 40 + 12 = 52.$$

8. If $u_{ij} \neq 0$ are components of a tensor of Type $(0,2)$ and the equation $f u_{ij} + g u_{ji} = 0$ holds, show that either $f = g$ and u_{ij} is skew symmetric or $f = -g$ and u_{ij} is symmetric.

$\Rightarrow$ we write, $f u_{ij} = -g u_{ji}$

or $\quad f^2 u_{ij} = -f g u_{ji}$

$$= -g(f u_{ji})$$

$$= -g(-g u_{ij}) = g^2 u_{ij}$$

or, $\quad f^2 u_{ij} - g^2 u_{ij} = 0$

or, $\quad (f^2 - g^2) u_{ij} = 0$

or, $\quad f^2 - g^2 = 0$, since u_{ij} be the component of covariant tensor; $u_{ij} \neq 0$.

$\therefore$ either $f = g$; or $f = -g$

If $f = g$, then $u_{ij} = -u_{ji}$

i.e u_{ij} is skew symmetric.

If $f = -g$, then $u_{ij} = u_{ji}$

i.e u_{ij} is symmetric.

9. If $g_{ij} = 0$ for $i \neq j$ and $g_{ij} \neq 0$ for $i = j$ show that $g^{ii} = \frac{1}{g_{ii}}$ (no summation)

$\Rightarrow$ Since $g_{ij} = 0$ for $i \neq j$, it follows that

$$\det g_{ij} = \begin{vmatrix} g_{11} & 0 & 0 & \cdots & 0 \\ 0 & g_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g_{nn} \end{vmatrix}$$

$$= g_{11}\, g_{22} \cdots g_{nn}$$

Now, $$g^{ii} = \frac{\text{cofactor of } g_{ii} \text{ in } \det g_{ij}}{\det g_{ij}}$$

$$= \frac{g_{11}\, g_{22} \cdots g_{i-1\, i-1}\, g_{i+1\, i+1} \cdots g_{nn}}{g_{11}\, g_{22} \cdots g_{nn}}$$

$$= \frac{1}{g_{ii}}$$

10) If $\psi(x^1, x^2, \ldots, x^n)$ is a scalar, test whether $\partial^r \psi / \partial x^i \partial x^j$ is a tensor.

$\Rightarrow$ Repeat in 2013.

11) If the tensors a_{ij} and c_{ij} are symmetric and u^i, v^i are the components of contravariant vectors such that $(a_{ij} - K c_{ij}) u^i = 0$ and $(a_{ij} - K' c_{ij}) v^i = 0$, where $K \neq K'$, prove that $a_{ij} u^i v^i = 0$.

$\Rightarrow$ we have,

$$(a_{ij} - Kc_{ij})u^i = 0 \quad \text{--- i)}$$

$$(a_{ij} - K'c_{ij})v^i = 0 \quad \text{--- ii)}$$

Multiplying (i) by v^j and (ii) by u^j and subtracting we get

$$a_{ij}u^iv^j - Kc_{ij}u^iv^j - a_{ij}u^jv^i + K'c_{ij}v^iu^j = 0.$$

interchanging i and j in the 2nd and fourth terms and noting that

$$c_{ij} = c_{ji} \quad \text{and} \quad a_{ij} = a_{ji}$$

we get, $a_{ij}u^iv^j - a_{ij}u^iv^j - Kc_{ij}u^iv^j + K'c_{ji}v^ju^i = 0$

$$\text{or,} \quad -(K - K')c_{ij}u^iv^j = 0$$

$$\because K \neq K' \quad c_{ij}u^iv^j = 0 \quad \text{--- iii)}$$

Again, multiplying (i) by v^j and remembering (iii) we get,

$$a_{ij}u^iv^j - \alpha 0 = 0$$

$$\text{or,} \quad a_{ij}u^iv^j = 0.$$

10) Define a zero vector and a null vector in a Riemannian space. Test whether a vector with components $(-1, 0, 0, 1/c^2)$, c being constant, in a space with line element

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^4(dx^4)^2$$ is a null vector.

$\Rightarrow \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^4.$

$$g_{ij} = 0 \quad , \text{ for } i \neq j$$

Let us consider the vector $(-1, 0, 0, 1/c^2)$

Then $$g_{ij} A^i A^j = g_{11} A^1 A^1 + g_{22} A^2 A^2 + g_{33} A^3 A^3 + g_{44} A^4 A^4$$

$$= (-1) \cdot (-1) \cdot (-1) + (-1) \cdot 0 \cdot 0 + (-1) \cdot 0 \cdot 0 + (c^4) \cdot \frac{1}{c^2} \cdot \frac{1}{c^2}$$

$$= -1 + 1 = 0 .$$

Hence the vector $(-1, 0, 0, 1/c^2)$ is null vector.

11.) If A^i and B^i are orthogonal vectors each of length l, then prove that

$$(g_{hj} g_{ki} - g_{hk} g_{ji}) A^h B^j A^k B^i = -l^4 .$$

$\Rightarrow$ Since A^i and B^i are orthogonal vectors each of length l we have,

$$g_{ij} A^i A^j = l^2$$

$$g_{ij} B^i B^j = l^2$$

and $$g_{ij} A^i B^j = 0 .$$

Now, $(g_{hj} g_{ki} - g_{hk} g_{ji}) A^h B^j A^k B^i$.

$$= g_{hj} g_{ki} A^h B^j A^k B^i - g_{hk} g_{ji} A^h B^j A^k B^i$$

$$= (g_{hj} A^h B^j)(g_{ki} A^k B^i) - (g_{hk} A^h A^k)(g_{ji} B^j B^i)$$

$$= 0 \times 0 - l^2 \times l^2$$

$$= -l^4$$