Various methods of solution

$$Pdx + Qdy + Rdz = 0$$

To solve any pfaffian equation $Pdx+Qdy+Rdz=0$ first of all we verify that the condition

$$P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=0$$

and then proceed to find the corresponding integral by applying any one of the following methods.

Method – I :- Solution by Inspection

In many problems it may be possible to re-arrange the terms in such a manner that the equation can be put in exact form and then just by inspection we get the solution of the problem.

Solve

① $(ydx + xdy)(a-z) + xy\,dz = 0.$

Soln :- $(ydx + xdy)(a-z) + xy\,dz = 0$

$\Rightarrow d(xy)(a-z) = -xy\,dz$

$$\Rightarrow \frac{d(xy)}{xy} = \frac{dz}{(z-a)}$$

integrating we get

$\log|xy| = \log|z-a| + \log|c|$ (c = arbitrary constant)

$$\Rightarrow \frac{xy}{z-a} = c$$

$\Rightarrow xy = c(z-a).$

Note: you have to satisfy 1st the condition of integrability.

(2) $x(y^2-a^2)dx + y(x^2-z^2)dy - z(y^2-a^2)dz = 0$

Soln: comparing the given ~~eqn~~ diff. & equation with $Pdx + Qdy + Rdz = 0$ we get

$P = x(y^2-a^2)$; $Q = y(x^2-z^2)$; $R = -z(y^2-a^2)$

now $\frac{\partial P}{\partial z} = 0$ | $\frac{\partial Q}{\partial x} = 2xy$ | $\frac{\partial R}{\partial x} = 0$

$\frac{\partial P}{\partial y} = 2xy$ | $\frac{\partial Q}{\partial z} = -2yz$ | $\frac{\partial R}{\partial y} = -2yz$

now, $P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$= \cancel{x(y^2-a^2)(-y+2yz) + y(x^2-z^2)(}$

$= x(y^2-a^2)(-2yz+2yz) + y(x^2-z^2)(0-0)$

$+ (-z)(y^2-a^2)(2xy - 2xy)$

$= 0$.

Hence the condition of integrability is satisfied.

now $x(y^2-a^2)dx + y(x^2-z^2)dy - z(y^2-a^2)dz = 0$

Dividing each term by $(y^2-a^2)(x^2-z^2)$ we get

$$\frac{x\,dx}{x^2-z^2} + \frac{y\,dy}{y^2-a^2} - \frac{z\,dz}{x^2-z^2} = 0,$$

$$\Rightarrow \frac{x\,dx - z\,dz}{x^2-z^2} + \frac{y\,dy}{y^2-a^2} = 0$$

$$\Rightarrow \frac{2x\,dx - 2z\,dz}{x^2-z^2} + \frac{2y\,dy}{y^2-a^2} = 0$$

$$\Rightarrow \frac{d(x^2-z^2)}{(x^2-z^2)} + \frac{d(y^2-a^2)}{y^2-a^2} = 0$$

Integrating bothside we get

$$\log|x^2-z^2| + \log|y^2-a^2| = \log|c|$$

(c = arbitrary constant)

$$\Rightarrow \log|(x^2-z^2)(y^2-a^2)| = \log|c|$$

$$\Rightarrow (x^2-z^2)(y^2-a^2) = c.$$

(3) Solve

$$\frac{yz}{x^2+y^2}dx - \frac{xz\,dy}{x^2+y^2} - \tan^{-1}(y/x)\,dz = 0.$$

Here $P = \frac{yz}{x^2+y^2}$; $Q = \frac{-xz}{(x^2+y^2)}$; $R = -\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial P}{\partial z} = \frac{y}{(x^2+y^2)}$$

$$\frac{\partial P}{\partial y} = z\left[\frac{(x^2+y^2) - y \times 2y}{(x^2+y^2)^2}\right]$$

$$\frac{\partial P}{\partial y} = \frac{(x^2-y^2)z}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial z} = \frac{-x}{(x^2+y^2)}$$

$$\frac{\partial Q}{\partial x} = -z\left[\frac{(x^2+y^2) - x \times 2x}{(x^2+y^2)^2}\right]$$

$$= \frac{-z(y^2-x^2)}{(x^2+y^2)^2}$$

$$= \frac{z(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial R}{\partial x} = -\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{\frac{x \times 0 - y}{x^2}}{(x^2+y^2)/x^2} = \frac{y}{(x^2+y^2)}$$

$$\frac{\partial R}{\partial y} = -\frac{\frac{\partial}{\partial y}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$$

$$= -\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = -\frac{x^2}{x(x^2+y^2)} = \frac{-x}{(x^2+y^2)}$$

now,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{yz}{(x^2+y^2)}\left(\frac{-x}{x^2+y^2} + \frac{x}{x^2+y^2}\right) - \frac{xz}{(x^2+y^2)}\left(\frac{y}{x^2+y^2} - \frac{y}{x^2+y^2}\right)$$
$$- \tan^{-1}\left(\frac{y}{x}\right)\left(\frac{(x^2-y^2)z}{(x^2+y^2)^2} - \frac{z(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$= 0.$

Hence the condition of integrability is satisfied.

now equation can be written as,

$$\frac{z(y\,dx - x\,dy)}{(x^2+y^2)} - \tan^{-1}\left(\frac{y}{x}\right) dz = 0$$

$$\Rightarrow z\left[\frac{\frac{y\,dx - x\,dy}{x^2}}{1+\left(\frac{y}{x}\right)^2}\right] = \tan^{-1}\left(\frac{y}{x}\right) dz$$

$$\Rightarrow z\,\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \tan^{-1}\left(\frac{y}{x}\right) dz$$

$$\Rightarrow \frac{d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}{\tan^{-1}\left(\frac{y}{x}\right)} = \frac{dz}{z}.$$

now integrating bothsides we get,

$$\Rightarrow \log\left|\tan^{-1}\left(\frac{y}{x}\right)\right| = \log|z| + \log|c|$$

c = arbitrary constant.

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = zc$$

$$\Rightarrow \frac{y}{x} = \tan(cz)$$

$$\Rightarrow y = x\tan(cz)$$

which is required solution.

H.T

1. $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2z\,dz = 0$

   Ans:- $(x + y + z^2) = ce^{-x^2}$.

2. $(yz + xyz)\,dx + (zx + xyz)\,dy + (xy + xyz)\,dz = $

   Ans:- $xyz = ce^{-(x+y+z)}$.

3. $(y+z)\,dx + dy + dz = 0$

   Ans:- $x + \log(y+z) = c$.

(Here c is arbitrary constant)