SEM-IV
PHYH-C IX: ELEMENTS OF MODERN PHYSICS

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Quantum Mechanical Operators

Quantum Mechanical Operators

An operator is a mathematical rule that changes a given function into a new function.

If A is an operator, it is generally represented by $\hat{A}$

$\Longrightarrow$ If $\hat{A}$ is an operator and stands for the operator $\frac{\partial}{\partial x}$, then when it operates on a function x^2 it gives

$$\frac{\partial}{\partial x}(x^2) = 2x$$

In quantum mechanics, each dynamical variables such as position momentum energy are represented by linear operators. They commute with a constant or with the sum of the two linear operators but their product may or may not be commutative.

$\Longrightarrow$ A linear operator has the following defining property:

- If $\hat{\alpha}$ is a linear operator, c is a constant and $f(x)$ is a function, then

$$\hat{\alpha}\{cf(x)\} = c\hat{\alpha}f(x) \tag{1}$$

i.e. $\hat{\alpha}$ commutes with an arbitrary constant c.

• Linear operator s obey distributive law: Thus if $f_1(x)$ and $f_2(x)$ are two arbitrary functions and $\hat{\alpha}$ is a linear operator then

$$\hat{\alpha}\{f_1(x) + f_2(x)\} = \hat{\alpha} f_1(x) + \hat{\alpha} f_2(x) \tag{2}$$

As an example , if $\hat{\alpha} f(x) = \frac{d^2}{dx^2}$ then
$\frac{d^2}{dx^2}\{f_1(x) + f_2(x)\} = \frac{d^2}{dx^2} f_1(x) + \frac{d^2}{dx^2} f_2(x)$

• Linear operators obey associative laws: Sums and products of linear operators are also linear operators. The sum of two linear operators is commutative. But the product of two linear operators may or may not be commutative.

■ If $\hat{\alpha}$ and $\hat{\beta}$ are two operators, the commutator bracket of these two operator represented by $[\hat{\alpha}\ ,\hat{\beta}]$ and is defined as

$$[\hat{\alpha}\ ,\hat{\beta}] = \hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha} \tag{3}$$

• If $[\hat{\alpha}\ ,\hat{\beta}] = 0$ i.e.$\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha}$, the two vectors are called commutative.

• If $[\hat{\alpha}\ ,\hat{\beta}] \neq 0$ i.e.$\hat{\alpha}\hat{\beta} \neq \hat{\beta}\hat{\alpha}$, the two vectors are called non-commutative.

prob. − 1. Find the value of $[\hat{x},\ \frac{\hat{\partial}}{\partial x}]$

prob. − 2. Show that the operators $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are commutative.

prob.3. What is the value of $[\frac{\hat{\partial}}{\partial x}, \quad \frac{\hat{\partial}}{\partial t}]$?

prob.4. Prove that$[\hat{x}, \quad \hat{p_x}] = i\hbar$. Hence give its physical significance.

Prob − 1. Let $f(x)$ be function on which these two operators are operate;

Thus

$$[\hat{x}, \ \frac{\hat{\partial}}{\partial x}]f(x)$$
$$= [x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x]f(x)$$
$$= x\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}(xf(x))$$
$$= x\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial x} - f(x)$$
$$= -f(x)$$
$$\text{Hence } [\hat{x}, \ \frac{\hat{\partial}}{\partial x}]f(x) = -f(x)$$
$$[\hat{x}, \ \frac{\hat{\partial}}{\partial x}] = -1$$

Prob. − 2. Let $f(x)$ be a function on which these two operator operate

$$[\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}]f(x)$$
$$= [\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x}]f(x)$$
$$= \frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}f(x) - \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x}f(x)$$
$$= \frac{\partial^3 f}{\partial x^3} - \frac{\partial^3 f}{\partial x^3} = 0$$
$$\text{So } [\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}] = 0$$

Therefore, the given two operators are commutative.

$Prob-3$ Let f be a function of x and t on which these two operators will operate.

$$[\frac{\hat{\partial}}{\partial x}, \quad \frac{\hat{\partial}}{\partial t}]f$$
$$= [\frac{\partial}{\partial x}\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\frac{\partial}{\partial x}]f$$
$$= \frac{\partial}{\partial x}(\frac{\partial f}{\partial t}) - \frac{\partial}{\partial t}(\frac{\partial f}{\partial x})$$
$$= \frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 f}{\partial t \partial x}$$
$$[\frac{\hat{\partial}}{\partial x}, \quad \frac{\hat{\partial}}{\partial t}] = 0$$

Hence, the two operators are commutative.

$Prob-4$ Let $f(x)$ be a function on which these two operator operate

$$[\hat{x}, \quad \hat{p_x}]f(x)$$
$$= [\hat{x}\hat{p_x} - \hat{p_x}\hat{x}]f(x)$$
$$= x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}\{xf(x)\}$$
$$= [\hat{x}, \quad \hat{p_x}]f(x) = -\frac{\hbar}{i}f(x)$$

So

$$[\hat{x}, \quad \hat{p_x}] = -\frac{\hbar}{i} \tag{4}$$

Physical Significance

This equation implies that any co-ordinate and his conjugate momentum does not commute.Thus it is impossible to measure position and momentum simultaneously along same direction with same accuracy. This supports the Heisenberge's position momentum uncertainty principle.