2.a.(i) If (X,ρ) is a metric space and $\sigma(x,y) = \dfrac{\rho(x,y)}{1+\rho(x,y)}$, $x,y \in X$ then show that σ is also a metric on X such that ρ and σ are induce the same topologies on X. (4)

Ans: For $x,y \in \mathbb{R}$, $\sigma(x,y) = \dfrac{\rho(x,y)}{1+\rho(x,y)} \geq 0$, $\because \rho(x,y) \geq 0$, ρ is a metric on X.

$= 0$, iff $x=y$

Again $\sigma(x,y) = \dfrac{\rho(x,y)}{1+\rho(x,y)} = \dfrac{\rho(y,x)}{1+\rho(y,x)}$ [$\because \rho$ is symmetric]

$= \sigma(y,x)$.

For $x,y,z \in \mathbb{R}$.

$$\sigma(x,y)+\sigma(y,z) = \frac{\rho(x,y)}{1+\rho(x,y)} + \frac{\rho(y,z)}{1+\rho(y,z)}$$

$$\geq \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)}$$

$$= \frac{\rho(x,y)+\rho(y,z)}{1+\rho(x,y)+\rho(y,z)}$$

$$= \frac{1}{\frac{1}{\rho(x,y)+\rho(y,z)}+1}$$

$$\geq \frac{1}{1+\frac{1}{\rho(x,z)}} \quad [\because \rho(x,z) \leq \rho(x,y)+\rho(y,z)]$$

$$= \frac{\rho(x,z)}{1+\rho(x,z)}$$

$$= \sigma(x,z).$$

i.e. $\sigma(x,z) \leq \sigma(x,y)+\sigma(y,z)$.

Triangle inequality holds. Hence, σ is a metric on X.

2.a.(ii) In a metric space (X,ρ) let $E \subset X$, define the closure of E and show that the closure of E is a closed set in (X,ρ) (2)

Proof: Def^n: Let B be a subset of (X,d) and let B' (derive set of B) denotes the set of all its limit points. Then the subset $B \cup B'$ is called the closure of B and its denoted by $\bar{B}$. i.e. $\bar{B} = B \cup B'$.

If $E=\varphi$ then $\bar{E}=\varphi$, which is closed.
If $E=X$, then $\bar{E}=X$, " " ".

Now in order to prove that $\bar{E}$ is closed we need to show that $(\bar{E})^c = X \setminus \bar{E}$ is open. In other words, for all $x \in (\bar{E})^c$, $\exists$ a (+ve) real $r=r(x)$ such that open sphere $S(x,r)$ with centre x lies entirely within $(\bar{E})^c$, i.e. $x \in S(x,r) \subset (\bar{E})^c$ i.e. we are to show that $S(x,r) \cap \bar{E} = \phi$

Now, $x \in (\bar{E})^c \Rightarrow x \notin \bar{E} = E \cup E' \Rightarrow x \notin E$ as well as $x \notin E'$

$\Rightarrow x \notin E$ as well as x is not a limit point of the set E in (X,ρ)

$\Rightarrow \exists\ r(>0)$ such that $S(x,r) \cap [E \setminus \{x\}] = \phi$

$\Rightarrow S(x,r) \cap E = \phi$, Since $x \notin E$ ($x \notin E \Rightarrow E = E \setminus \{x\}$)

$\Rightarrow S(x,r)$ contains no points of the set E.

$\Rightarrow S(x,r) \subset E^c$.

Now we need to show that $S(x,r)$ does not contains any limit point of the set E. we see $z \in S(x,r) \Rightarrow \rho(x,z) < r$.

Let, $r' = r - \rho(x,z) > 0$. Then $S(z,r') \subset S(x,r)$.

So, $S(z,r') \cap E \subset S(x,r) \cap E = \phi$

i.e. $S(z,r') \cap E = \phi$

$\Rightarrow S(z,r')$ contains no limit point of the set E, so z can not be a limit point of the set E.

Combining the above results we conclude that we can find $r(>0)$ such that ~~$x \notin A \cup A' = \bar{A} \Rightarrow x \in$~~

$x \notin E \cup E' = \bar{E} \Rightarrow S(x,r) \cap \bar{E} = \phi$

$\Rightarrow x \in S(x,r) \subset (\bar{E})^c$.

Since x is arbitrary, it follows that $(\bar{E})^c$ is open.

Consequently $\bar{E}$ is closed.

5. a. Let (X,ρ) and (Y,σ) be metric spaces and let $f: X \to Y$ be a function.

(i) Show that f is continuous at $x_0 \in X$. iff for every sequence $\{x_n\}$ in X which converges to x_0 the sequence $\{f(x_n)\}$ in Y converges to $f(x_0)$.

Proof: 1st assume that the function $f: (X,\rho) \to (Y,\sigma)$ is continuous at a point $x_0 \in X$. we are to prove that

$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$. as $n \to \infty$.

Let, $\epsilon > 0$ be arbitrary choosen. Since f is continuous at the point x_0 there exist a $\delta(>0)$ such that $\rho(x_n,x_0) < \delta$

$\Rightarrow \sigma(f(x_n), f(x_0)) < \epsilon$

Since, $x_n \to x$ as $n \to \infty$ in (X,ρ), corresponding to $\delta > 0$, $\exists$ a natural no. N depending on δ such that $n > N \Rightarrow \rho(x_n,x) < \delta$.

Combining the above two results we conclude that

$n > N \Rightarrow \sigma(f(x_n), f(x_0)) < \epsilon$.

where N is a natural no. depending on δ and hence depend on ϵ.

This implies ~~the sufficient part~~ $\{f(x_n)\}$ converges to $f(x_0) \in Y$.

To prove the sufficient part we are to show that if for all sequences $\{x_n\}$ converging to x_0 in (X, ρ) the corresponding sequences $\{f(x_n)\}$ converges to $f(x_0)$ in (Y, σ), then f is continuous at x_0.

Let us assume, if possible f is not continuous at the point x_0. Then $\exists$ at least one $\varepsilon > 0$ such that $\forall \delta\ (>0)$

$$\rho(x^*, x_0) < \delta \quad \text{but} \quad \sigma(f(x), f(x_0)) \geq \varepsilon.$$

For at least one $x \in X$.

We take, in particular a sequence of δ's given by $\delta = \frac{1}{n}$, $n \in \mathbb{N}$

So, corresponding to each natural no. n, $\exists\ x_n \in X$ such that

$$\rho(x_n, x_0) < \frac{1}{n} \quad \text{but} \quad \sigma(f(x_n), f(x_0)) \geq \varepsilon.$$

$\Rightarrow$ $f(x_n) \not\to f(x)$ in (Y, σ) although $x_n \to x_0$ as $n \to \infty$ in (X, ρ)

— which is a contradiction to our hypothesis.

Hence, f must be continuous at the point x.

5.a.(ii) When is the function f said to be uniformly continuous? Show that if X is compact and f is continuous in X then f is uniformly continuous on X. (1+5=6)

Ans: Def^n: Let, $f : (X, d) \to (Y, \rho)$ is said to be uniformly continuous if given $\varepsilon > 0$ there is a '+ve' δ such that $\rho(f(x_1), f(x_2)) < \varepsilon$, whenever $d(x_1, x_2) < \delta$.

Let, (X, d) be a compact metric space and (Y, ρ) be any metric space, and let $f : (X, d) \to (Y, \rho)$ be continuous function.

If $\varepsilon > 0$, $x \in X$ by continuity of f at x, we find a '+ve' $\delta(x)$ such that $\rho(f(x'), f(x)) < \varepsilon/2$, whenever $d(x', x) < \delta(x)$.

Now consider the family of open balls $B\{x, \frac{1}{2}\delta(x)\}_{x \in X}$.

Clearly this family $B\{x, \frac{1}{2}\delta(x)\}_{x \in X}$ is an open cover for X.

But X is compact so there is a finite subcover $B\{x_1, \frac{1}{2}\delta(x_1)\}$, $B\{x_2, \frac{1}{2}\delta(x_2)\}, \ldots, B\{x_n, \frac{1}{2}\delta(x_n)\}$.

Let us choose δ such that $\delta < \min_{1 \leq i \leq n} \{\frac{1}{2}\delta(x_i)\}$

Let, $u, v \in X$ with $d(u,v) < \delta$.

Let, $u \in B(x_i, \frac{1}{2}\delta(x_i))$, for some i $[1 \leq i \leq n]$

Then $d(v, x_i) \leq d(v, u) + d(u, x_i)$

$< \delta + \frac{1}{2}\delta(x_i)$

$< \frac{1}{2}\delta(x_i) + \frac{1}{2}\delta(x_i)$

$< \frac{1}{2}\delta(x_i)$

$\therefore$ $\rho(f(u), f(v)) \leq \rho(f(u), f(x_i)) + \rho(f(x_i), f(v))$

$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Hence, the theorem.

5. b. (i) Let, $C[a,b]$ be the class of all continuous functions on $[a,b]$ with its metric ρ defined by $\rho(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)|$, $f \in C[a,b]$. Prove that $C[a,b]$ is complete. (4)

Proof: We 1st show that $C[a,b]$ is a metric space. w.r.to 'sup' metric. [verify this]

Let, $\{f_n\}$ be a Cauchy sequence of $C[a,b]$ w.r.to the sup metric ρ.

i.e. $\rho(f_n, f_m) \to 0$ as $n, m \to \infty$.

or, $\sup_{a \leq x \leq b} |f_n(x) - f_m(x)| \to 0$ as $n, m \to \infty$.

So for given $\varepsilon > 0$, $\exists$ an index N, satisfing $\sup_{a \leq x \leq b} |f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq N$.

So, $|f_n(x) - f_m(x)| \leq \sup_{a \leq x \leq b} |f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq N$ and $\forall x \in [a,b]$.

i.e. $|f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq N$ and $x \in [a,b]$.

Thus the real sequence $\{f_n(x)\}$ is a Cauchy sequence and by Cauchy general Principle of convergence, we have $\lim_n f_n(x)$ exist as a unique member.

Define $f: [a,b] \to \mathbb{R}$ by the formula $f(x) = \lim_n f_n(x)$, $a \leq x \leq b$.

Now show that $f \in C[a,b]$.

Now from $\sup_{a \leq x \leq b} |f_n(x) - f_m(x)| < \varepsilon$, for $n, m \geq N$.

Now taking $m \to \infty$ and n fixed $(\geq N)$, we have $\sup_{a \leq x \leq b} |f_n(x) - f_m(x)|$ $\leq \varepsilon$, for $n \geq N$. or, $|f_n(x) - f(x)| \leq \varepsilon$, for $n \geq N$ and, $\forall x \in [a,b]$.

This shows that $\{f_n\}$ converges to f uniformly over $[a,b]$. So by Weierstrass theorem f is continuous over closed interval $[a,b]$.

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i.e. $f \in c[a,b]$.

Now from $|f_n(x) - f(x)| \leq \varepsilon$, for $n \geq N$ and $\forall\, x \in [a,b]$.

We have $\sup_{a \leq x \leq b} |f_n(x) - f(x)| \leq \varepsilon$, $\forall\, n \geq N$.

or, $\rho(f_n, f) \leq \varepsilon$, $\forall n \geq N$.

i.e. $\lim_n f_n = f$ in $c[a,b]$.

and hence $(c[a,b], \rho)$ is a complete metric space.

5.b.(ii) State and Prove Cantor Intersection Theorem for a complete metric space. (2+6=8)

Ans:

Statement : A necessary and sufficient condition that a metric space (X,d) is complete is that every decreasing chain of non-empty closed sets $\{F_n\}$ with $\text{Diam}(F_n) \to 0$ as $n \to \infty$ has $\bigcap_{n=1}^{\infty} F_n$ as a single element.

Proof : Let, (X,d) be a complete metric space. and $\{F_n\}$ be a decreasing chain of non-empty closed subset in X with $\text{Diam}(F_n) \to 0$ as $n \to \infty$.

Let, $x_n \in F_n$. Now for any p, a '+ve' integer $x_{n+p} \in F_{n+p}$.

Clearly, $x_{n+p} \in F_n$ [$\because F_n \supset F_{n+p}$].

So, $d(x_n, x_{n+p}) \leq \text{diam}(F_n) \to 0$ as $n \to \infty$.

Hence, $\{x_n\}$ is a Cauchy sequence of (X,d).

By Compactness of (X,d), $\{x_n\}$ is convergent.

i.e. $\lim_n x_n = u \in X$.

We have $x_{n+p} \in F_n$. Keeping n fixed and letting $p \to \infty$, we get

$\lim_{p \to \infty} x_{n+p} = u$.

Since, F_n is closed, we have $u \in F_n$.

Now as n varis, we have $u \in \bigcap_{n=1}^{\infty} F_n$.

If $v \in \bigcap_{n=1}^{\infty} F_n$, then u and $v \in F_n$, for each n.

So, $d(u,v) \leq \text{Diam}(F_n) \to 0$ as $n \to \infty$.

i.e. $d(u,v) \leq 0$, i.e. $d(u,v) = 0$. [$\because d(u,v) \geq 0$].

i.e. $u = v$. i.e. $\bigcap_{n=1}^{\infty} F_n = \{u\}$

Conversely, Suppose the condition holds in a metric space (X,d) and we show that (X,d) is complete.

Let, $\{x_n\}$ be a Cauchy sequence of elements in (X,d).

Let us put $H_n = \{x_n, x_{n+1}, x_{n+2}, \dots, x_{n+p}, \dots\}$, $n = 1,2,3,\dots$

We see that $\{H_n\}$ forms a decreasing chain of non-empty subsets of (X,d).

Hence, $\{\overline{H_n}\}$ is also decreasing chain of non-empty closed subset of (X,d) such that $Diam(\overline{H_n}) = Diam(H_n) \to 0$ as $n \to \infty$.

Hence, by assumption, we have $\bigcap_{n=1}^{\infty} \overline{H_n}$ consist of a single member say $\{u\}$, clearly u and $x_n \in \overline{H_n}$

So, $d(u, x_n) \le Diam(\overline{H_n}) \to 0$ as $n \to \infty$.

i.e. $\lim_n d(u, x_n) = 0$.

i.e. $\lim x_n = u \in X$.

i.e. $\{x_n\}$ is convergent and $\lim_n x_n = u \in X$.

This shows that (X,d) is a complete metric space.

5.c. (i) Let, E be a collection of connected sets in a metric space (X,ρ). If $\cap\{E : E \in E\}$ is non-empty then prove that $\cup\{E : E \in E\}$ is connected. (6)

Ans: Let us assume, if possible the set E is not connected.

Then E must has disconnection (P,Q) i.e. we can express $E = P \cup Q$, where $P \ne \phi$, $Q \ne \phi$, $\overline{P} \cap Q = \phi$ and $P \cap \overline{Q} = \phi$

Now since P and Q are non-empty seperated subsets of $E = \cup\{E_\alpha : \alpha \in \Delta\}$, [where E_α be a collection of connected sets in a metric space (X,d)]

there should exists i and $j \in \Delta$ such that $E_i \subset P$, $E_j \subset Q$

If possible, let us assume $\exists$ no $i \in \Delta$ such that $E_i \subset P$.

Since, $E_i \subset E$ and $E = P \cup Q$. It follows that $E_i \cap Q \ne \phi, \forall i \in \Delta$

Similarly, $\exists$ no $j \in \Delta$ satisfing $E_j \subset Q$, we must have $E_j \cap P \ne \phi$, $\forall j \in \Delta$.

Let us choose perticular $i \in \Delta$.

Then $E_i \cap P \ne \phi$ as well as $E_i \cap Q \ne \phi$. Since P and Q are separated and $E_i \cap P$ and $E_i \cap Q$ are non-empty subsets of P and Q. we have, $E_i \cap P$ and $E_i \cap Q$ are two non-empty seperated sets.

Therefore, E_i must be disconnected as $E_i = (E_i \cap P) \cup (E_i \cap Q)$ — this contradicts our assumption that E_i is connected.

Since, P, Q are seperated, E_i and E_j are also separated (clearly $i \neq j$) and hence are disjoint. Consequently, $\cap\{E_i : i \in \Delta\}$ $= \phi$, which is a contradiction to our hypothesis.

Hence, the set $\cup\{E : E \in \mathcal{E}\}$ must be connected.

5.c.(ii) Let, (X, ρ) and (Y, σ) be two metric spaces and let $f: X \to Y$ be a ~~continuous~~ function. If $E \subset X$ is connected and if f is continuous on E then prove that $f(E)$ is connected in Y. (6)

Ans: Let, $f: X \to Y$ be continuous.

We are to prove that for any connected set $E \subset X$, $f(E)$ is connected in (Y, σ).

case I. If $E = \phi$, $f(E) = f(\phi) = \phi$. Hence $f(E)$ is connected.

case II If E is a singleton set, $E = \{a\}$ (say), then $f(E) = \{f(a)\}$ is also singleton set. In this case $f(E)$ is connected.

case III. Let, $E \subset X$ contains atleast two points. We are to prove that $f(E)$ is connected in (Y, σ).

If possible let $f(E)$ is disconnected, then $\exists$ two nonempty sets G_1 and G_2 open in (Y, σ) such that $f(E) \subset G_1 \cup G_2$, where $f(E) \cap G_1 \neq \phi$, $f(E) \cap G_2 \neq \phi$. But $f(E) \cap (G_1 \cap G_2) = \phi$.

Since, G_1 and G_2 are open in the metric space (Y, σ), and f is continuous mapping, then $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are open in (X, ρ).

Also, $E \cap (f^{-1}(G_1) \cap f^{-1}(G_2)) = f^{-1}(f(E)) \cap \{f^{-1}(G_1) \cap f^{-1}(G_2)\}$

$$= f^{-1}(f(E) \cap (G_1 \cap G_2))$$

$$= f^{-1}(\phi)$$

$$= \phi.$$

Since, $f(E) \subset G_1 \cup G_2$, it follows that $E \subset f^{-1}(G_1 \cup G_2)$

$$= f^{-1}(G_1) \cup f^{-1}(G_2).$$

Finally, $f(E) \cap G_1 \neq \phi \Rightarrow f^{-1}(f(E) \cap G_1) \neq \phi$

$\Rightarrow E \cap f^{-1}(G_1) \neq \phi$

Similarly, $f(E) \cap G_2 \neq \phi \Rightarrow f^{-1}(f(E) \cap G_2) \neq \phi$

$\Rightarrow E \cap f^{-1}(G_2) \neq \phi$.

Thus, we can express $E = E_1 \cup E_2$, where $E_1 = E \cap f^{-1}(G_1) \neq \phi$ and $E_2 = E \cap f^{-1}(G_2) \neq \phi$ But, $E_1 \cap E_2 = \phi$

Consequently, E is disconnected in (X, d), which is a contradiction. Since, E is connected.
Hence, $f(E)$ must be connected in (Y, σ). This complete the proof.

5. d. (i) Let, (X, ρ) and (Y, σ) be metric spaces and let $f: X \to Y$ be continuous. If X is compact prove that the image $f(X)$ is also compact in Y. (6)

Proof: Let, $\{B_\alpha\}$ be an open cover of $f(X)$ in Y, then let us put $B_\alpha = f(X) \cap H_\alpha$, where H_α is open set in (Y, σ).
So consider new open sets $\{H_\alpha\}_{\alpha \in \Delta}$ in (Y, σ).
Since, f is continuous, so $f^{-1}(H_\alpha)$, for each $\alpha \in \Delta$ is an open set in (X, ρ) such that $f(X) \subseteq \bigcup_{\alpha \in \Delta} B_\alpha = \bigcup_{\alpha \in \Delta} (f(X) \cap H_\alpha)$
Since, $\{f^{-1}(H_\alpha)\}_{\alpha \in \Delta}$ becames an open cover in X.
So by compactness of X, we have a finite subcover for X say $\{f^{-1}(H_1)\}, \{f^{-1}(H_2)\}, \dots \{f^{-1}(H_n)\}$, so that we have

$$X \subseteq f^{-1}(H_1) \cup f^{-1}(H_2) \cup \dots \cup f^{-1}(H_n.$$

$$\text{or, } f(X) \subseteq H_1 \cup H_2 \cup \dots \cup H_n.$$

$$\text{or, } f(X) \subseteq \{H_1 \cap f(X)\} \cup \{H_2 \cap f(X)\} \cup \{\dots \cup \{H_n \cap f(X)\}.$$

$$= B_1 \cup B_2 \cup \dots \cup B_n.$$

Hence, $\{B_1, B_2, \dots, B_n\}$ becomes a finite subcover for $f(X)$.
and hence, $f(X)$ is compact.

5d(ii) Let, $\mathbb{R}$ be the set of real numbers and let $\sigma(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$, $x, y \in \mathbb{R}$. Show that σ is a metric on $\mathbb{R}$ and that the metric space $(\mathbb{R}, \sigma)$ is not complete.

Ans: for any $x, y \in \mathbb{R}$, $\sigma(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| \geq 0$

$= 0$ iff $\frac{x}{1+|x|} = \frac{y}{1+|y|}$

For any $x, y \in \mathbb{R}$, $\sigma(x,y) = \sigma(y,x)$. is trivial. iff $x = y$.

If for $x, y, z \in \mathbb{R}$, $\sigma(x,y) + \sigma(y,z) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| + \left|\frac{y}{1+|y|} - \frac{z}{1+|z|}\right|$

$$\geq \left|\frac{x}{1+|x|} - \cancel{\frac{y}{1+|y|}} + \cancel{\frac{y}{1+|y|}} - \frac{z}{1+|z|}\right|$$

$$= \left|\frac{x}{1+|x|} - \frac{z}{1+|z|}\right|.$$

$= \sigma(x,z)$.

i.e. $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

Triangle inequality holds.

Hence, σ is a metric on the set of all real no.'s.

Let us consider the sequence $\{x_n\}$ in $\mathbb{R}$. where $x_n = n, \forall n \in \mathbb{N}$. Corresponding to any preassigned $\varepsilon > 0$, choose a $K \in \mathbb{N}$ such that $K > \frac{2}{\varepsilon} - 1$. Then $\sigma(x_m, x_n) = \left|\frac{x_m}{1+|x_m|} - \frac{x_n}{1+|x_n|}\right| = \left|\frac{m}{1+m} - \frac{n}{1+n}\right|$

$$= \left|\frac{m}{1+m} - 1 + 1 - \frac{n}{1+n}\right|$$

$$= \left|-\frac{1}{1+m} + \frac{1}{1+n}\right|$$

$$\leq \frac{1}{m+1} + \frac{1}{n+1}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \forall m,n \geq K.$$

Thus $\{x_n\}$ is a Cauchy Sequence in $(\mathbb{R}, \sigma)$.

Now to show that $\{n\}$ is not convergent sequence, we assume to the contrary.

If Possible, let $\{n\}$ be a convergent sequence converging to some $x \in \mathbb{R}$. Then $\sigma(x_n, x) \to 0$ as $n \to \infty$. in the real line.

Case-I. $x = 0$ i.e. $\sigma(n,0) \to \left|\frac{n}{1+n}\right| = \frac{n}{1+n} \to 0$ which is impossible.

Case-II. $x < 0$, say $x = -y$ $(y > 0)$.

Then $\sigma(n,x) = \left|\frac{n}{1+n} - \frac{y}{1+y}\right| \to 0$, which is again impossible.

Since, $\left\{\frac{n}{1+n}\right\}$ is an increasing sequence.

Case-III. $x > 0$, then $\sigma(n,x) = \left|\frac{n}{1+n} - \frac{x}{1+x}\right| \to 0$ which is again impossible as in case II.

Hence, $\{n\}$ cannot be a convergent sequence.

Hence, $(\mathbb{R}, \sigma)$ is not complete.

1.a. Find the diameter of the set $\{(x,y) : 0 < x < 1 ; y = e^x\}$. w.r.to the usual metric on $\mathbb{R}^2$.

Ans: $\left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$ is a sequence in $(0,1)$ such that $\frac{1}{n+1} \to 0$ as $n \to \infty$. Now, the function $(: \mathbb{R} \to \mathbb{R}) x \to e^x$ is continuous on $\mathbb{R}$. So the sequence $\left\{e^{\frac{1}{n+1}} : n \in \mathbb{N}\right\}$ is continuous on $\mathbb{R}$. so the sequence $\left\{e^{\frac{1}{n+1}} : n \in \mathbb{N}\right\}$ converges to $e^0 = 1$ in $\mathbb{R}$.

Thus $\{(\frac{1}{n+1}, e^{\frac{1}{n+1}}) ; n \in \mathbb{N}\}$ is a sequence in A, which converges to $(0,1)$ in $\mathbb{R}^2$. So, $(0,1) \in \bar{A}$, Similarly, $(1, e) \in \bar{A}$.

Now, $\text{diam}(A) = \text{diam}(\bar{A}) \geq d((0,1),(1,e)) = \sqrt{(0-1)^2 + (1-e)^2}$

$= \sqrt{1+e^2-2e+1}$

$= \sqrt{e^2-2e+2} \longrightarrow (i)$

Again, $\sup\{d((x_1,y_1),(x_2,y_2)) : (x_1,y_1), (x_2,y_2) \in A\}$.

$= \sup\{[(x_1-x_2)^2 + (e^{x_1}-e^{x_2})^2]^{1/2} : (x_1,y_1), (x_2,y_2) \in A\}$.

$\leq [(\sup\{|x_1-x_2| : 0<x_1,x_2<1\})^2 + (\sup\{|e^{x_1}-e^{x_2}| : 0<x_1,x_2<1\})^2]^{1/2}$

$= [1^2 + (e-1)^2]^{1/2}$ [$\because$ both the function x and e^x are m.i. on $\mathbb{R}$]

$= \sqrt{e^2-2e+2}$

Thus $\text{Diam}(A) \leq \sqrt{e^2-2e+2} \longrightarrow (ii)$

$\therefore$ From (i) and (ii), we get

$\text{Diam}(A) = \sqrt{e^2-2e+2}$.

(d) Show that the set of rational no. in $[0,1]$ is not complete in $\mathbb{R}$?

Ans: Let, X = the set of rational no. in $[0,1]$.
Consider a sequence $\{x_n\}$, where $x_n = \frac{1}{2^n}$, $n=1,2,\ldots$

Here, $d(x_n,x_m) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| \leq \frac{1}{2^n} + \frac{1}{2^m} \to 0$ as $n,m \to \infty$.

Also, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$

$\therefore \{x_n\}$ is cauchy and converges to a number of space ~~[0,1]~~ X

Again consider a sequence $\{x_n\}$, where $x_n = (1+\frac{1}{n})^n$, $n=1,2,3,\ldots$

Here, $\{x_n\}$ is cauchy but $\lim x_n = \lim (1+\frac{1}{n})^n = e \notin$ ~~[0,1]~~ X.

Thus ~~the~~ X is not complete metric space.

e. Examine where the set $\{(x,y) : x=0, 0<y<1\} \cup \{(x,y) : 0<x<1; y=0\}$ is connected in $\mathbb{R}^2$?

Ans: Let, $X = C \cup D$, where

$C = \{(x,y) : x=0, 0<y<1\}$ and $D = \{(x,y) : 0<x<1, y=0\}$.

$C \neq \emptyset$ and $D \neq \emptyset$

But $C \cap D = \emptyset$

Hence, X is connected set in $\mathbb{R}^2$.

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