SEM-IV

PHYH-C IX: ELEMENTS OF MODERN PHYSICS

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Eigen function, Eigen value, Expection value

Eigen function: When an operator acting on a function will always produce the same function multiplied by a constant factor , the function is called an eigen function and the constant is known as the eigen value of the given operator.

Thus, if $\hat{A}$ is an operator that operates on a given function $f(x)$

$$\hat{A}f(x) = \lambda f(x) \tag{1}$$

where λ is a constant. This equation is called eigen value equation, the constant λ known as eigen value and the function $f(x)$ is called the eigen function of the corresponding operator $\hat{A}$. All the operators in quantum mechanics have an eigen function and eigen values.

Example: Let the operator $\hat{\alpha} = \frac{d^2}{dx^2}$ and $\phi(x) = e^{-2x}$

Then

$$\hat{\alpha}\phi(x) = \lambda\phi(x)$$

$$\frac{d^2}{dx^2}(e^{-2x}) = 4(e^{-2x}) \tag{2}$$

Thus operating the function e^{-2x} by the operator $\frac{d^2}{dx^2}$ multiplies the function by the numerical factor 4. We may say that e^{-2x} is an eigen function of the operator $\frac{d^2}{dx^2}$ belonging to the eigen value 4.

To be an eigen function, $\phi(x)$ must be well behaved which means that itmust satisfy the following requirement:

(a) $\phi(x)$ must be single valued every where;

(b) $\phi(x)$ must be square integrable so that integral of its modulus square is finite;

(c) $\phi(x)$ must be continuous everywhere;

(d) the derivative $\frac{d\phi}{dx}$ should be continuous everywhere;

(e) $\phi(x)$ must remain finite or vanish as $x \to \pm\infty$.

Problem-1 Which one of the following functions are eigen functions of the operator $\frac{d^2}{dx^2}$? .Calculate also the eigen value where appropriate;
$i.\ Cosx,\quad ii\ \ e^{ix}$,$iii.\ \ Sin^2x,\ iv.\ \ e^{4x}$

Expectation value

The expectation value of a dynamical quantity is the mathematical expectation for the result of a single measurement. In otherwards, it is average of the results of a large number of measurements on independent identical system having identical functions.

The average or expectation value of any function $f(x)$ is given by

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t) f(x) \psi(x,t) dx \tag{3}$$

Subject to the normalization condition

$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)dx = 1$ i. e . $\psi(x,t)$ is normalized.

Expectation value of momentum $\hat{p_x} = -i\hbar\frac{\delta}{\delta x}$

$$
\begin{aligned}
\langle p_x \rangle &= \int_{-\infty}^{\infty} \psi^*(x,t)\hat{p_x}\psi(x,t)dx \\
&= \int_{-\infty}^{\infty} \psi^*(x,t)\hat{p_x}\psi(x,t)dx \\
&= \int_{-\infty}^{\infty} \psi^*(x,t)(-i\hbar\frac{\delta}{\delta x})\psi(x,t)dx \\
&= -i\hbar\int_{-\infty}^{\infty} \psi^*(x,t)\frac{\delta}{\delta x}\psi(x,t)dx
\end{aligned}
$$

Expectation value of Energy

$$
\begin{aligned}
\langle E \rangle &= \int_{-\infty}^{\infty} \psi^*(x,t)(i\hbar\frac{\delta}{\delta t})\psi(x,t)dx \\
\langle E \rangle &= i\hbar\int_{-\infty}^{\infty} \psi^*(x,t)\frac{\delta}{\delta t}\psi(x,t)dx
\end{aligned}
$$

Problem-2 A particle is moving along X axis has the wave function

$$
\psi(x) = \begin{cases} bx, & \text{between } \ x = 0 \ \ and \ \ x = 1 \\ 0 & \qquad\quad \text{elsewhere} \end{cases}
$$

Find the expectation value $\langle x \rangle$ of the particles position.

If $\psi(x)$ is the wave function of the moving particle, expectation values of position is

$$
\begin{aligned}
\langle x \rangle &= \textstyle\int_{-\infty}^{\infty} \psi^*(x)(\hat{x})\psi(x)dx \\
&= \textstyle\int_0^1 x|\psi(x)|^2 dx \\
&= \textstyle\int_0^1 x|bx|^2 dx \\
&= b^2 \textstyle\int_0^1 x^3 dx \\
&= \tfrac{b^2}{4}
\end{aligned}
$$

Problem-3 Find the expectation values $\langle x \rangle$ and $\langle p_x \rangle$ for the Gaussian packet given by
$\psi(x) = (\frac{1}{\sigma\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{x^2}{2\sigma^2}} e^{(ikx)}$