Ring Theory

■ **Def^n**: A non-empty set R is said to form a ring w.r.to two binary composition, addition $(+)$ and multiplication $(\cdot)$ defined on it if the following conditions are satisfied.

(i) $(R,+)$ is a commutative group.

- $a+b \in R$ (closure property)
- $a+(b+c) = (a+b)+c$ (Associative)
- $a+0 = 0+a = a$ (Identity)
- $a+(-a) = (-a)+a = 0$ (Inverse)
- $a+b = b+a$, $\forall a,b,c \in R$. (Commutative).

(ii) $(R,\cdot)$ is a semi-group.

- $a\cdot b \in R$
- $a\cdot(b\cdot c) = (a\cdot b)\cdot c$, $\forall a,b,c \in R$.

(iii) For any three elements $a,b,c \in R$

$a\cdot(b+c) = a\cdot b + a\cdot c$ (Left distributive law)

$(b+c)\cdot a = b\cdot a + c\cdot a$ (Right distributive law)

– This ring is denoted by $(R,+,\cdot)$ or simply by R.

NOTE

(i) A ring $(R,+,\cdot)$ is called a commutative ring if the multiplication is commutative. i.e. $a\cdot b = b\cdot a$, $\forall a,b \in R$

(ii) The additive identity element in R is called zero element in R.

(iii) A multiplicative identity (if exists) is known as unity element of the ring R and it is denoted by I.

(iv) Multiplicative inverse of non-zero element if exist is known as unit element of the ring R.

■ Results :

In a ring $(R, +, \cdot)$

(i) $a \cdot 0 = 0 \cdot a = 0$, $\forall a \in R$

(ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$, $\forall a, b \in R$

(iii) $(-a) \cdot (-b) = a \cdot b$, $\forall a, b \in R$.

Proof : (i) We have,

$$(0 + 0) = 0$$

$$\Rightarrow a \cdot (0 + 0) = a \cdot 0$$

$$\Rightarrow a \cdot 0 + a \cdot 0 = a \cdot 0 \quad \text{[Left distributive law]}$$

$$\Rightarrow (-a \cdot 0) + (a \cdot 0 + a \cdot 0) = (-a \cdot 0) + (a \cdot 0)$$

$$\Rightarrow (-a \cdot 0 + a \cdot 0) + a \cdot 0 = 0 \quad [\because \text{Addition is associative}]$$

$$\Rightarrow 0 + a \cdot 0 = 0$$

$$\Rightarrow a \cdot 0 = 0$$

Similarly, $(0 + 0) = 0$

$$\Rightarrow (0 + 0) \cdot a = 0 \cdot a$$

$$\Rightarrow 0 \cdot a + 0 \cdot a = 0 \cdot a \quad \text{[Right distributive law]}$$

$$\Rightarrow (0 \cdot a + 0 \cdot a) + (-0 \cdot a) = (0 \cdot a) + (-0 \cdot a)$$

$$\Rightarrow 0 \cdot a + ((0 \cdot a) + (-0 \cdot a)) = 0$$

$$\Rightarrow 0 \cdot a + 0 = 0$$

$$\Rightarrow 0 \cdot a = 0$$

Hence, $a \cdot 0 = 0 \cdot a = 0$ (Proved)

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(ii) We have, $[b + (-b)] = 0$

$$\therefore a \cdot [b + (-b)] = a \cdot 0 = 0$$

By left distributive law.

$$a \cdot b + a \cdot (-b) = 0.$$

$-(ab) \in R$, Adding $-(a \cdot b)$ to both side, we have

$$-(a \cdot b) + [(a \cdot b) + a \cdot (-b)] = -(a \cdot b) + 0.$$

or, $[-(a \cdot b) + (a \cdot b)] + a \cdot (-b) = -(a \cdot b).$

or, $0 + a \cdot (-b) = -(a \cdot b)$

or, $a \cdot (-b) = -(a \cdot b)$

~~Again $a + (-a) = 0$.~~

~~Therefore, $[a + (-a)] \cdot b = 0 \cdot b = 0$.~~

Similarly, $(-a) \cdot b = -(a \cdot b)$

Combining these two, we get

$$a \cdot (-b) = (-a) \cdot b = -(a \cdot b).$$

(iii) Let, $p = -a$, then $p \in R$.

$$\begin{aligned}(-a) \cdot (-b) &= p \cdot (-b) \\ &= -(p \cdot b) \\ &= -[(-a) \cdot b] \\ &= -[-(a \cdot b)] \\ &= a \cdot b\end{aligned}$$

This complete the proof.