SEM-IV
PHYH-C IX: ELEMENTS OF MODERN PHYSICS

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Probability and probability current densities in one dimension

We have seen that the wave function $\psi(x,t)$ which describes the complete space-time behavior of a particle in one dimensional motion has amplitudes in those regions where the particle is likely to be found with grater probability.
We shall assume that the quantity

$$|\psi(x,t)|^2 dx = \psi^*(x,t)\psi(x,t)dx$$

is proportional to the probability of finding the particle in the interval $x$to$x+dx$ at the time t Where ψ^* is the complex conjugate of ψ.
The total probability of finding the particle

$$P = \int_{-\infty}^{\infty} |\psi(x,t)|^2 dx$$

Position probability density as

$$\rho(x,t) = \frac{|\psi(x,t)|^2}{\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx}$$

Hence total probability will be

$$P = \int_{-\infty}^{\infty} P(x,t)dx = \frac{\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx}{\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx} = 1$$

i.e. the total probability must be unity. Since $|\psi(x,t)|^2$ is necessarily positive.

If $\psi(x,t)$ is multiplied by a complex constant N such that $\psi_N(x,t) = N\psi(x,t)$ where $\psi_N(x,t)$ satisfies the relation

$$\int_{-\infty}^{\infty} |\psi_N(x,t)|^2 dx = |N|^2 \int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$$

where $\psi_N(x,t)$ is said to be the normalized wavefunction.
where

$$|N|^2 = \frac{1}{\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx}$$

The above relations can be generalized for the case of three dimensional motion

$$\rho(r,t) = |\psi(r,t)|^2 = \psi * (r,t)\psi(r,t)$$

$$P = \int_{\tau} |\psi r, t|^2 d\tau = 1$$

where $\psi(r,t)$is here regarded as normalized and the integration carried out over the entire three dimensional space.

Probability current density
Since the total probability

$$\int_{\tau} \rho(r,t) d\tau = \int_{\tau} |\psi r, t|^2 d\tau = 1 = \text{constant}$$

at every instant of time any decrease of probability in a given volume element $d\tau$ must be associated with the corresponding increase of probability in some other element.The situation is analogusto the flow of change from a given volume element. The in the total quantity of charge contained in the given volume element should be equal to the net flow of charge through the surface enclosing the given volume element which is expressed by the equation of continuity

$$\frac{\delta\rho}{\delta t} + \nabla.J = 0 \tag{1}$$

where ρ is the charge density and J is the current density.

Considering a finite volume element τ enclosed by the surface S we calculate the rate of change of probability finding the particle in τ

$$\frac{\delta}{\delta t}\int_\tau \rho(r,t)d\tau = \frac{\delta}{\delta t}\int_\tau \psi^*(r,t)\psi(r,t)d\tau \tag{2}$$

$$= \int_\tau (\frac{\delta\psi^*}{\delta t}\psi + \psi^*\frac{\delta\psi}{\delta t})d\tau \tag{3}$$

Now *psi* satisfies the Schrodinger equation

$$H\psi(r,t) = i\hbar\frac{\delta}{\delta t}\psi(r,t) \tag{4}$$

were H explicitly

$$\{-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\}\psi(r,t) = i\hbar\frac{\delta}{\delta t}\psi(r,t) \tag{5}$$

assuming V is real, so complex conjugate of above equation

$$\{-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\}\psi^*(r,t) = -i\hbar\frac{\delta}{\delta t}\psi^*(r,t) \tag{6}$$

so we have

$$\frac{\delta\psi}{\delta t} = \frac{1}{i\hbar}\{-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\}\psi \tag{7}$$

$$\frac{\delta\psi^*}{\delta t} = -\frac{1}{i\hbar}\{-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\}\psi^* \tag{8}$$

Then

$$\frac{\delta\psi^*}{\delta t}\psi + \psi^*\frac{\delta\psi}{\delta t}$$

$$= \frac{i}{\hbar}[(-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*)\psi - \psi^*(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi)]$$

$$= -i\frac{\hbar^2}{2m}(\psi\nabla^2\psi^* - \psi * \nabla^2\psi) \tag{9}$$

we know from vector analysis that u and v are two scalar , we can write

$$\nabla.(u\nabla v) = u\nabla^2 v + \nabla u.\nabla v$$
$$\nabla.(v\nabla u) = v\nabla^2 u + \nabla v.\nabla u$$
$$=\text{v } \nabla^2 u + \nabla u.\nabla v$$
$$\nabla.(u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

so we can write

$$\psi\nabla^2\psi^* - \psi^*\nabla^2\psi = \nabla.(\psi\nabla\psi^* - \psi^*\nabla\psi) \tag{10}$$

the from equation $2, 3 and 10$

$$\frac{\delta}{\delta t}\int_\tau \rho d\tau = -\frac{i\hbar}{2m}\int_\tau \nabla.(\psi\nabla\psi^* - \psi^*\nabla\psi)d\tau \tag{11}$$

$$\Rightarrow \frac{\delta\rho}{\delta t} + \nabla.J = 0 \tag{12}$$

we define probability current density

$$J = \frac{i\hbar}{2m}((\psi\nabla\psi^* - \psi^*\nabla\psi) \tag{13}$$

For one dimensional motion, the probability current density

$$J_x = \frac{i\hbar}{2m}((\psi\frac{\delta\psi^*}{\delta t} - \psi^*\frac{\delta\psi}{\delta t}) \tag{14}$$

Problem-1 Find the probability density and probability current density for the Gaussian packet given by
$\psi(x) = (\frac{1}{\sigma\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{x^2}{2\sigma^2}} e^{(ikx)}$