SEM-IV
PHYH-C IX: ELEMENTS OF MODERN PHYSICS

Manoj Kumar Saha,
Assistant Professor, Department of Physics K C College Hetampur

Concept of Phase velocity, Group velocity and relation between them

When a progressive wave of amplitude a propagating with velocity v along the positive direction of X axes can be written as

$$y = aSin\omega(t - \frac{x}{v}) \tag{0.1}$$

Where ω is the angular frequency . Usually, this wave velocity v is known as phase velocity v_p.

from eq (0.15) the phase ϕ of the wave at any time t at position x

$$\phi(x,t) = \ omega(t - \frac{x}{v}) \tag{0.2}$$

Differentiating both sides of the equation

$$\frac{d\phi}{dt} = \omega(1 - \frac{1}{v}\frac{dx}{dt}) \tag{0.3}$$

For constant phase

$$\frac{d\phi}{dt} = 0 \tag{0.4}$$

i.e

$$\omega(1 - \frac{1}{v}\frac{dx}{dt}) = 0 \tag{0.5}$$

or

$$v = (\frac{dx}{dt})_\phi \tag{0.6}$$

= the velocity with which a displacement having a constant phase move forward. This is called phase velocity v_p.
So eq (0.15) can be written in the form of phase velocity v_p as

$$y = aSin\omega(t - \frac{x}{v_p}) = aSin(\omega t - \frac{\omega}{v_p}x) = aSin(\omega t - kx) \tag{0.7}$$

Where $k = \frac{\omega}{k}$ is the propagation constant of the wave.

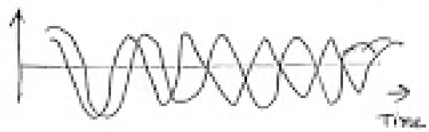
So **phase velocity** $v_p = \frac{\omega}{k}$

Group Velocity:

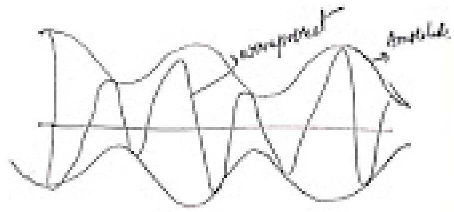
Group velocity : [v_g]

The superposition of several waves will form a complex wave called a wave group or wavepacket. In a wavepacket, the different component moves with different phase velocity but whole group advances through the medium with a constant velocity. This constant velocity with which a group of waves travel through the medium is called group velocity.

The energy of the group of waves is transported with group velocity through the medium.



Two interfering waves.



Group of waves.

Let us now consider that a wave group arises from the combination of two harmonics waves with ~~slit~~ slightly different angular frequency $\Delta\omega$ and wave number Δk.

$$\therefore \quad y_1 = a\sin(\omega t - kx)$$

$$y_2 = a\sin[(\omega t + \Delta\omega t) - (k + \Delta k)x]$$

So the resultant displacement

$$y = y_1 + y_2 = a\sin(\omega t - kx) + a\sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

$$= 2a\cos\left(-\frac{\Delta\omega}{2} + \frac{\Delta k}{2}x\right)\sin\left[\left(\frac{2\omega + \Delta\omega}{2}\right)t - \left(\frac{2k + \Delta k}{2}\right)x\right]$$

$$\because \sin c + \sin d = 2\cos\frac{c-d}{2}\cdot\sin\frac{c+d}{2}$$

Now since $\Delta\omega$ & Δk are small compared to ω & k we can neglect $\Delta\omega$ & Δk

$$\therefore \boxed{y = 2a \cos\left(\frac{\Delta\omega}{2} t - \frac{\Delta k}{2} x\right) \sin(\omega t - kx)}$$

This equation represent that the resultant wave is an amplitude modulated wave which is travelling with phase velocity $V_p = \frac{\omega}{k}$.

Again the amplitude of the resultant group of waves varies with time with circular frequency $\frac{\Delta\omega}{2}$ is

$$A = 2a \cos\left(\frac{\Delta\omega}{2} t - \frac{\Delta k}{k} x\right)$$

at $x = 0$, $t = 0$ maximum value of amplitude

$$R_{max} = 2a$$

$\therefore$ The group velocity

$$V_g = \lim_{\Delta k \to 0} \frac{\Delta\omega/2}{\Delta k/2}$$

$$= \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

So the group velocity $V_g = \frac{d\omega}{dk}$

Relation

we know, the phase velocity $v_p = \frac{\omega}{k}$

$\therefore \omega = k v_p$

Now, group velocity $v_g = \frac{d}{dk}(\omega)$

$$= \frac{d}{dk}(k v_p)$$

$$= v_p + k \frac{dv_p}{dk}$$

$$= v_p + k \frac{dv_p}{d\lambda} \cdot \frac{d\lambda}{dk}$$

Since $k = 2\pi/\lambda$

$$dk = -\frac{2\pi}{\lambda^2} d\lambda$$

$$\therefore \frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi}$$

$$= -\frac{1}{2\pi} \times \frac{4\pi^2}{k^2}$$

$$\therefore \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

$$\therefore v_g = v_p + k \frac{dv_p}{d\lambda} \cdot \left(-\frac{2\pi}{k^2}\right)$$

$$\boxed{v_g = v_p - \lambda \frac{dv_p}{d\lambda}}$$ ⟹ This is the relation between phase velocity & group velocity.

⟹ 1. For dispersive medium $v_g > v_p$ group velocity is greater than phase velocity.

⟹ 2. For non-dispersive medium, the waves of all wavelengths moves with same velocity, $\frac{dv_p}{d\lambda} = 0$, $\therefore \boxed{v_p = v_g}$

This is valid for electromagnetic waves in vacuum and ~~electromagnetic~~ the elastic wave in a homogeneous medium.

⟹ For light wave in vacuum $\frac{dv_p}{d\lambda} = 0$ $\therefore \boxed{v_p = v_g = c}$